Optimal lot-sizing, quality improvement and inspection errors for multistage production systems

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We develop multistage lot-sizing models for imperfect production processes. The effect of inspection errors in screening non-conforming items at various stages has been incorporated. Inspection and restoration of the processes at all stages have also been considered. Numerical examples are presented to illustrate various aspects of the models developed. It is shown that when inspection and restoration are introduced, lower costs are incurred and larger lot sizes are produced because of reductions in quality control costs.

1. Introduction

The classical economic production quantity (EPQ) model has been extensively studied in the literature under various conditions (Silver et al. 1998). Several practical considerations have been incorporated into the original model. In particular, the effect of deteriorating processes on the EPQ in single-stage production systems received a lot of attention from researchers. Rosenblatt and Lee (1986) studied the effect of substandard quality, due to a deteriorating process, on lot-sizing decisions. In particular, they showed that the optimal production cycle is shorter than that of the classical EPQ model for the single-stage lot-sizing problem. The issue of the effect of imperfect processes has been addressed by many authors including Porteus (1986), Lee and Rosenblatt (1987, 1989) and Groenevelt et al. (1992), among others. Rahim (1994) and Rahim and Ben-Daya (1996) looked at the effect of EPQ on the economic design of a $\bar{x}$-control chart. They developed an integrated model for the inventory and quality control problems for a class of deteriorating processes where the in-control period follows a general probability distribution with increasing hazard rate.

However, integrating quality aspects in the EPQ model has not been adequately addressed in the context of multistage production systems. Imperfect production processes present additional practical complications that are not present in perfect systems. The following important issues must be addressed when dealing with imperfect multistage production processes.

- Non-conforming items must be screened so that they are not passed to subsequent stages to avoid the waste of resources and unnecessary processing.
- While screening non-conforming items, errors may be committed. Non-conforming items may be incorrectly accepted and good items incorrectly rejected.

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Processes shift to an out of control state where non-conforming items are produced must be prevented or at least delayed, and detected quickly if it occurs.

Ben-Daya and Rahim (1997) developed a lot-sizing model for a multistage system that addresses the first two issues. However, there is a need to build a quality improvement aspect into this model. This paper addresses the third issue and develops multistage models that take into account inspection errors and process inspection and restoration for imperfect processes. Numerical examples are presented to illustrate important aspects of the proposed model. In particular, it is shown that inspection and restoration of the processes at various stages lead to lower expected total cost due to reductions in quality-related costs.

This paper is organized as follows. In section 2 the multistage production system is described and the necessary notations are developed. The mathematical model incorporating process deterioration and inspection errors is presented in section 3. This model is extended in section 4 by incorporating process inspection and restoration. Numerical examples and sensitivity analysis are presented in section 5. Finally, section 6 contains a summary of the paper and some conclusions.

2. Multistage production system

Consider a multistage production system producing a single end product in \( n \) stages. It is assumed that demand and production rates at all stages are constant and the production rate at any stage exceeds the demand rate, i.e. \( P_j > D, j = 1, 2, \ldots, n \). No backlogging is permitted. Beyond the first stage, the work-in-process (WIP) at any subsequent stage is characterized by a single semifinished item, until it becomes the final product after the last production stage.

The inventory items in the multistage system are arranged in a hierarchical manner. The final product comprises the topmost level (level 1), intermediate products at various prior stages occupy the subsequent levels, and raw materials and other inputs lie at the lowest level (level \( n + 1 \)). The first manufacturing stage, \((n + 1)\)th level items are processed to obtain the WIP at level \( n \), and so on, until at stage \( n \) the level 2 semifinished item is transformed to the final product. Each unit of production is passed to the next stage as soon as processing at the current stage is completed without waiting for the completion of the whole batch. These production systems are not uncommon in practice (Banerjee and Burton 1990). Consequently, in the presence of perfect processes and no inspection errors, a uniform lot size of \( Q \) units is processed at any production stage.

It is assumed that at each production stage the process starts in the in-control state producing items of perfect or acceptable quality. However, the process may shift to the out-of-control state after a random time with known probability distribution. Once in the out-of-control state, the process starts producing a fixed percentage of non-conforming items.

First, the necessary notations are presented.

General notation

\[ n \quad \text{number of stages}, \]
\[ Q \quad \text{production lot size for the end product}, \]
\[ Q_j \quad \text{production lot size for the product at the } j\text{th level}, \]
\[ D \quad \text{demand rate of the end product (units/unit time)}, \]
$P_j$ production rate (units/unit time) at which the $(j + 1)$th level item is converted to the $j$th level inventory item,

$A_j$ set-up cost at the $j$th level,

$r$ fractional inventory carrying cost ($$/$/unit time),

$C_j$ production cost excluding set-up at the $j$th level,

$I_j$ expected average inventory level at the $j$th level,

$t_j$ production time per cycle at the $j$th level,

ETC expected total cost per cycle.

**Notation related to quality**

$s_j$ unit cost of producing a non-conforming item at the $j$th level,

$\alpha_j$ fraction of non-conforming units at the $j$th level,

$N_j$ expected number of non-conforming items produced at the $j$th level,

$F_j$ process shift distribution at the $j$th level,

$F_j$ probability density function of the time to shift at the $j$th level.

**Notation related to inspection errors**

$E_{1,j}$ probability of incorrectly rejecting a conforming item at the $j$th level,

$E_{2,j}$ probability of incorrectly accepting a non-conforming item at the $j$th level,

$\pi_{1,j}$ cost of incorrectly rejecting a conforming item at the $j$th level,

$\pi_{2,j}$ cost of incorrectly accepting a non-conforming item at the $j$th level.

**Notation related to process inspection and restoration**

$\tau$ detection delay, i.e. the time elapsed between the occurrence of a shift and the end of the production cycle at any stage,

$R(\tau)$ restoration cost, which is function of the detection delay: $R(\tau) = r_o + r_1\tau$,

$\nu_j$ inspection cost of the process at level $j$.

3. **Lot-sizing models with imperfect production processes**

It is assumed that the quality of the output of the various stages is not perfect. On each cycle at the $j$th level, the production process may shift at a random time to the out-of-control state and starts producing a fixed fraction $\alpha_j$ of non-conforming items. The following practical consideration will be incorporated in the model.

- Processed items at stage $j$ are inspected and put in the inventory to be used when necessary at stage $j - 1$. (One does not wait until the entire lot $Q_j$ is produced to start production at the next stage.) Non-conforming items are screened before passing to the next production stage to avoid unnecessary processing.
- Inspection process is error prone. Both type I and type II errors may be committed. That is non-conforming items may be incorrectly accepted and conforming items may be incorrectly rejected.

A process may produce non-conforming items while in the in-control state. This is due to the items falsely accepted in previous stages. In practice, type II errors are small due to advances in inspection technology. Consequently, the number of such items is very small compared with the lot size produced. This justifies the assumption
made throughout this paper that non-conforming items are produced only when the process is in the out-of-control state. This issue will be discussed further in section 5.

A similar model was proposed by Ben-Daya and Rahim (1997). The purpose of this section is twofold. First, the model is presented so that the paper is self-contained. Second, we propose here a simpler recursive relationship formula for obtaining lot sizes at different levels.

The expected total cost consists of set-up costs, inventory carrying costs, quality-related costs due to production of non-conforming items, and additional costs due to inspection errors. These various costs are now derived.

3.1. Quality-related costs

To determine quality-related costs, we need to determine the expected number of non-conforming items produced at any stage once the corresponding process shifts to an out of control state. The expected number of non-conforming items produced at the $j$th level is given by:

$$N_j = \int_0^{t_j} \alpha_j P_j (t_j - t) f_j(t) \, dt,$$

where $t$ is the elapsed time for which the process at the $j$th level remains in the in-control state before a shift occurs, and $t_j$ is the length of the production run at level $j$. It is assumed that once a shift occurs, the process stays in the out-of-control state until the next process inspection. If, at the $j$th level, the time to shift distribution is exponential with mean $\frac{1}{\theta_j}$, i.e. $f_j(t) = \frac{1}{\theta_j} e^{-t/\theta_j}$, then

$$N_j = \alpha_j P_j (t_j - \theta_j + \theta_j e^{-t_j/\theta_j}),$$

where $t_j = \frac{Q_j}{P_j}, j = 1, 2, \ldots, n$ is the production cycle length at the $j$th level.

A lot of $Q$ conforming products are required for the final product. Non-conforming items at each level are removed and not passed to the following stage. While screening products at any stage to remove non-conforming items, if any, two types of errors may be committed:

- Rejecting a conforming item at the $j$th level, with probability $E_{1,j}$. This is known as type I error.
- Accepting a non-conforming item at the $j$th level, with probability $E_{2,j}$. This is known as type II error.

From figure 1, it can be seen that

$$Q_{j-1} = (Q_j - N_j)(1 - E_{1,j}) + N_j E_{2,j}$$

$$= Q_j (1 - E_{1,j}) - (1 - E_{1,j} - E_{2,j}) \alpha_j P_j \left[ \frac{Q_j}{P_j} - \theta_j + \theta_j e^{-Q_j/(P_j \theta_j)} \right].$$

The last equality follows from (2). Hence, the expected quality costs per unit time due to non-conforming items and inspection errors are given by:
3.2. Set-up and inventory holding cost

The expected set-up cost per unit time is simply

\[ E(SC) = \frac{D}{Q} \sum_{j=1}^{n} A_j. \]  

(6)

As to the inventory holding cost, let us first consider the case of two production stages and deal with the two cases \( P_1 \leq P_2 \) and \( P_1 \geq P_2 \), separately.

In this case the lot sizes \( Q_j \) at different levels are related by (4) and the expected number of non-conforming items that are actually rejected at level \( j \) is given by:

\[ N_j' = N_j(1 - E_{2,j}) + (Q_j - N_j)E_{1,j}, \]

(7)

which is the sum of correctly rejected non-conforming items and incorrectly rejected conforming items.
Case 1: $P_1 \leq P_2$. The average inventories at the three levels of the two-stage production systems are as follows (figure 2).

(i) Level 1: At this level the maximum inventory level $I_{1,\text{max}}$ is

$$I_{1,\text{max}} = (P_1 - D)t_1 - N'_{1}$$

$$= Q - \frac{D}{P_1}Q_1. \quad (8)$$

The last equality follows from that fact that $N'_{1}$ is given by (7) and that $Q = (Q_1 - N_1)(1 - E_{1,1}) + N_1E_{2,1}$ (figure 1). The inventory profile during the production time $t_1$ is built at a rate of $P_1 - D$. However, this rate is reduced once the process shifts to the out of control state. The inventory level reaches $I_{1,\text{max}}$ at time $t_1$. We approximate this behaviour over $t_1$ by a straight line from $(0,0)$ to $(t_1, I_{1,\text{max}})$. Using this assumption and since the length of the inventory cycle at level 1 is $Q/D$, the average inventory at level 1 is given by:

$$I_1 = \frac{1}{2} \left[ Q - \frac{D}{P_1}Q_1 \right]. \quad (9)$$

(ii) Levels 2 and 3: Using similar arguments it can be shown that

$$I_2 = \frac{DQ_1}{2Q} \left[ \frac{Q_1}{P_1} - \frac{Q_2}{P_2} \right] \quad (10)$$

$$I_3 = \frac{DQ_2}{2QP_2}. \quad (11)$$

Figure 2. Inventory time plots for a two-stage system; Case 1: $P_1 \leq P_2$. 
Case 2: \( P_1 \geq P_2 \). The average inventories at levels 1 and 3 are the same as in Case 1 but \( I_2 \) can be derived as follows.

From figure 3, note that the maximum inventory level, \( I_{2,\text{max}} \), is equal to the accumulated inventory during the time \( (t_2' - t_1) \) at a rate of \( P_2 \) minus the expected number of non-conforming items actually rejected during the time \( t_2 - t_1 \) taking into account inspection errors. Hence, since the total number of non-conforming items produced during the time \( t_2 - t_1 \) is \( N_{2,\text{t_1}} \) given by:

\[
N_{2,\text{t_1}} = \alpha_2 P_2 [t_2' - \theta_2 + \theta_2 e^{-t_2'/\theta_2}],
\]

then using (7),

\[
I_{2,\text{max}} = P_2(t_2 - t_1) - \alpha_2 P_2 [t_2' - \theta_2 + \theta_2 e^{-t_2'/\theta_2}] (1 - E_{1,2} - E_{2,2}) - P_2 t_2' E_{1,2}. \tag{13}
\]

Noting that

\[
I_2 = \frac{1}{2} \frac{I_{2,\text{max}} t_2}{Q/D},
\]

we obtain after simplification the following expression for \( I_2 \):

\[
I_2 = \frac{DQ_2}{2Q} \left\{ \left[1 - E_{1,2} - \alpha_2 (1 - E_{1,2} - E_{2,2}) \right] t_2' - \alpha_2 \theta_2 (1 - E_{1,2} - E_{2,2}) (e^{-t_2'/\theta_2} - 1) \right\}, \tag{14}
\]

where \( t_2' = t_2 - t_1 = Q_2 / P_2 - Q_1 / P_1 \).

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Figure 3. Inventory time plots for a two-stage system; Case 2: \( P_1 \geq P_2 \).
The above results can be generalized to the $n$-stage case as follows.

\[
I_1 = \frac{1}{2} \left[ Q - \frac{D}{P_1} Q_1 \right]
\]

\[
I_j = \begin{cases} 
\frac{DQ_j}{2Q} \{[1 - E_{1,j} - \alpha_j(1 - E_{1,j} - E_{2,j})]t'_j] & \text{if } P_j \leq P_{j-1} \\
-\alpha_j \theta_j(1 - E_{1,j} - E_{2,j})(e^{-t'/\theta_j} - 1) & \text{if } P_j \geq P_{j-1} 
\end{cases}
\]

\[
I_{n+1} = \frac{DQ_n^2}{2Q P_n},
\]

where $t'_j = t_j - t_{j-1} = Q_j/P_j - Q_{j-1}/P_{j-1}$.

Consequently, the expected inventory holding cost per unit time is given by:

\[
HC = r \left[ \sum_{j=1}^{n+1} C_j I_j \right].
\]

where $I_1, I_2, \ldots, I_{n+1}$ are given by (15–17).

The expected total cost per unit time is simply the sum of (5), (6) and (18), i.e.

\[
ETC_1 = E(QC) + E(SC) + E(HC).
\]

4. Process inspection and restoration

We develop here an extended model that takes into consideration process inspection and restoration as a mean of improving quality.

Each stage is inspected at regular intervals during a production run and if it is found to be out of control, necessary actions are taken to restore it to the in control state. Let $\eta_j$ be the number of inspections of the process at level $j$, which is a decision variable. Scheduling inspections at regular intervals has been shown to be optimal under certain conditions in cases of exponentially distributed time to shift to the out of control state (e.g. Lee and Rosenblatt 1989). The memoryless property of the exponential also provide an intuitive appeal for such an inspection policy. The restoration cost is assumed to be a linear function of the detection delay and restoration time is negligible. Detection delay is defined as the time elapsed from the time of the shift to the out-of-control state occurs until the time it is detected in the following inspection.

Before deriving the various costs involved, we need to determine the number of non-conforming items produced at each production stage. Since a process is restored to the in-control state whenever it is found to be out of control during an inspection, the expected number of non-conforming items will be different from what was determined in the previous model.

If $\eta_j$ is the number of inspections of the $j$th level process and the inspection intervals are of equal length, then the length of each inspection interval is given by:
\[ t_{i,j} = \frac{t_j}{\eta_j} = \frac{Q_j}{\eta_j P_j}, \]

where \( t_j = Q_j / P_j \) is the length of the production run at level \( j \).

Consider the \( i \)th inspection interval of the process at level \( j \). Because of the scheduled maintenance inspection at the end of each interval, the process begins each interval in the in-control state. Define \( t \) to be the elapsed time to a shift to the out-of-control state since the last inspection. Because of the memoryless property of the exponential distribution, \( t \) is again exponential with mean \( 1 / \theta_j \). The expected number of non-conforming items, \( N_{i,j} \), produced during the \( i \)th interval is given by:

\[
N_{i,j} = \int_{0}^{t_j/\eta_j} \alpha_j P_j \left( \frac{t_j}{\eta_j} - t \right) \frac{1}{\theta_j} e^{-t/\theta_j} \, dt \cdot \frac{1}{\theta_j} e^{-t_j/\theta_j} \, \text{dt}
\]

\[
N_{i,j} = \alpha_j P_j \left[ \frac{t_j}{\eta_j} - \theta_j + \theta_j e^{-t_j/(\eta_j \theta_j)} \right]
\]

The total expected number of non-conforming items produced during a complete production run of the process at level \( j \), \( N_j \), is then:

\[
N_j = \alpha_j P_j \left[ t_j - \eta_j \theta_j (1 - e^{-t_j/(\eta_j \theta_j)}) \right].
\]

The cost model of this section consists of the following.

- Set-up and inventory holding costs.
- Quality-related costs due to non-conforming items and item inspection errors.
- Maintenance process inspection and restoration costs.

The above cost are derived next.

4.1. Set-up and inventory holding costs

The set-up cost is still given by (6). As to the inventory holding cost, it can be derived as follows.

The expressions for \( I_1 \) and \( I_n \) will remain the same as those given by (15) and (17). This should be clear from figures 2 and 3. However, the expression for \( I_2 \), in a two-stage system, will change for the case \( P_1 \geq P_2 \). In this case, we need to determine the new expression for \( I_2 \). Then it can be generalized to the \( n \)-stage case, as discussed above.

From figure 3, note that we need to determine the expected number of non-conforming items produced during the time \( t'_{2} = t_2 - t_1 \), call it \( N_{2,t'_{2}} \).

Let \( i_2 \) be the smallest integer such that \( i_2 (t_2/\eta_2) \geq t'_2 \),

\[
\tau_2 = (i_2 - 1)(t_2/\eta_2)
\]

\[
\tau'_2 = t'_2 - \tau_2,
\]

then

\[
N_{2,t'_{2}} = (i_2 - 1) \alpha_2 P_2 \left[ \frac{t_2}{\eta_2} - \theta_2 + \theta_2 e^{-t_2/(\eta_2 \theta_2)} \right] + \alpha_2 P_2 \left[ \tau'_2 - \theta_2 + \theta_2 e^{-\tau'_2/\theta_2} \right]
\]

\[
= \alpha_2 P_2 \left[ t'_2 - \theta_2 (i_2 - (i_2 - 1)) e^{-t_2/(\eta_2 \theta_2)} - e^{-\tau'_2/\theta_2} \right].
\]
Again from figures 1 and 3 and (7), note that

\[
I_{2,\text{max}} = P_2 t_2' - [N_{2,t_2'}(1 - E_{2,2}) + (P_2 t_2' - N_{2,t_2'})E_{1,2}]
\]

\[
= P_2 t_2'(1 - E_{1,2}) - N_{2,t_2'}(1 - E_{1,2} - E_{2,2})
\]

\[
= P_2 \left( \frac{Q_2}{P_2} - \frac{Q_1}{P_1} \right) (1 - E_{1,2}) (1 - E_{1,2} - E_{2,2}).
\]

\[
\alpha_2 P_2 [t_2' - \theta_2 (i_2 - (i_2 - 1)e^{-t_2'(\eta \theta_2)} - e^{-t_2'/\theta_2})].
\]

Therefore

\[
I_2 = \frac{I_{2,\text{max}} t_2'}{2} \frac{D}{Q}
\]

\[
= \frac{DQ_2}{2Q} \left\{ \left( \frac{Q_2}{P_2} - \frac{Q_1}{P_1} \right) (1 - E_{1,2})
\right. 

\left. - (1 - E_{1,2} - E_{2,2}) \alpha_2 [t_2' - \theta_2 (i_2 - (i_2 - 1)e^{-t_2'(\eta \theta_2)} - e^{-t_2'/\theta_2})] \right\}.
\]

This can be generalized to the n-stage case as follows:

\[
I_1 = \frac{1}{2} \left[ Q - \frac{D}{P_1} Q_1 \right]
\]

\[
I_j = \left\{ \begin{array}{ll}
\frac{DQ_j}{2Q} \left\{ \left( \frac{Q_j}{P_j} - \frac{Q_{j-1}}{P_{j-1}} \right) (1 - E_{1,j}) - (1 - E_{1,j} - E_{2,j})\alpha_2 
\right. 

\left. - \left[ t_j' - \theta_j (i_j - (i_j - 1)e^{-t_j'(\eta \theta_j)} - e^{-t_j'/\theta_j}) \right] \right\} & \text{if } P_j \leq P_{j-1} \\
\frac{DQ_{j-1}}{2Q} \left[ \frac{Q_{j-1}}{P_{j-1}} - \frac{Q_j}{P_j} \right] & \text{if } P_j \geq P_{j-1}
\end{array} \right.
\]

\[
j = 2, \ldots, n
\]

\[
I_{n+1} = \frac{DQ_n^2}{2Q P_n},
\]

where \( t_j' = t_j - t_{j-1} = Q_j/P_j - Q_{j-1}/P_{j-1}, i_j \) is the smallest integer such that \( i_j(t_j/\eta_j) \geq t_2', \tau_j = (i_j - 1)(t_j/\eta_j), \) and \( t_j' = t_j - \tau_j. \) Therefore the expected inventory holding cost is given by

\[
r \sum_{j=1}^{n+1} C_j I_j,
\]

where \( I_j, j = 1, \ldots, n + 1 \) are given by (26–28).

4.2. Process inspection and restoration cost

The inspection cost \( E(\text{IC}) \) is given by:

\[
\frac{D}{Q} \sum_{j=1}^{n} (\eta_j - 1)\nu_j.
\]

As to the restoration cost, \( E(\text{RC}) \), it can be derived as follows.
Since it is assumed that the restoration cost depends on the detection delay, then
the expected restoration cost during the $i$th inspection interval of length $t_{i,j} = \frac{Q_j}{(\eta_j P_j)}$ of the process at level $j$, $E(RC_{ij})$, is given by:

$$E(RC_{ij}) = \int_0^{t_{i,j}} (r_{o,j} + r_{1,j}(t_{i,j} - t)) \frac{1}{\theta_j} e^{-t/\theta_j} \, dt$$

$$= \left( r_{o,j} + r_{1,j} \frac{Q_j}{\eta_j P_j} - r_{1,j} \theta_j \right) + (r_{1,j} \theta_j - r_{o,j}) e^{-Q_j/(\eta_j P_j)}. \quad (31)$$

Consequently, the expected restoration cost for the whole system is simply:

$$E(RC) = \frac{D}{Q} \sum_{j=1}^{n} \eta_j \left( r_{o,j} - r_{1,j} \theta_j \right) + r_{1,j} \frac{Q_j}{\eta_j P_j} + (r_{1,j} \theta_j - r_{o,j}) e^{-Q_j/(\eta_j P_j)} \right), \quad (32)$$

where $r_{o,j}$ and $r_{1,j}$ are cost parameters for the process at level $j$.

Let

$$E(IRC) = E(IC) + E(RC) \quad (33)$$

be the expected inspection and restoration cost, where $E(IC)$ and $E(RC)$ are given by (30) and (32), respectively.

The expected total cost per unit time is simply the sum of (6), (29), (5) and (33), i.e.

$$ETC1 = E(SC) + E(HC) + E(QC) + E(IRC). \quad (34)$$

5. Optimization procedure and numerical examples

The problem is then to determine the lot size and inspection schedules at all stages that minimize the expected total cost $ETC$. The pattern search technique of Hooke and Jeeves (1962) was used to minimize the cost function. The search starts with a local exploration in small steps around some starting point. If the exploration is a success, i.e. the cost reduces during local exploration, the step size grows; if the exploration is a failure, the step size is reduced. If a change of direction is necessary, the method starts all over again with a new pattern. The search is terminated when the step size is reduced to a prespecified value or when the number of iterations equals a predetermined value, whichever occurs first. However, due to the characteristics of the cost function, some modifications to the standard method have to be made to account for inherent integrality constraints on the number of inspections at each stage. To ensure that the search technique results in a global minimum, the cost function has to be unimodal. This has been upheld by extensive numerical experimentation. General results are difficult to obtain due to the complexity of the cost function. This algorithm was coded and run on a personal computer.

Consider a three-stage problem with data given in table 1. The optimal solutions for the above example using the two models developed in sections 3 and 4 (Models 1 and 2, respectively) are given in table 2.

Sensitivity analysis was also conducted with respect to fraction non-conforming and inspection errors. The results are summarized in tables 3 and 4, respectively.

Table 3 provides the results obtained for various inspection error levels for a system not subjected to process inspection and restoration. These results are obtained using the model developed in section 3. Table 4 provides the results obtained for the extended model developed in section 4 where the processes at all
stages are subjected to inspection and restoration, if necessary. Higher fractions lead to higher costs and smaller lot sizes, as expected. However, lot sizes are less sensitive to inspection errors.

Tables 3 and 4 show that the lot sizes at the three stages are more sensitive to the fraction of non-conforming items than they are to type I and type II errors. However, the expected total cost is sensitive to fraction non-conforming and errors. Because of the difference in non-conforming items produced, lot sizes are larger when process inspection and restoration are introduced. This is due to the improvement in quality rate. For example, without inspection and restoration and when $\alpha_j = 0.05$, $E_{1,j} = E_{2,j} = 0.01$, $j = 1, 2, 3$, the lot size of the final product is $Q = 812$ much less than the perfect lot size of 1254 units. When inspection and restoration are introduced, $Q = 1208$.

A comparison of these tables 3 and 4 also reveals other important aspects of the proposed model. In particular, for same level of inspection errors and fractions of non-conforming items, the proposed model yields lower costs. For example for $\alpha_j = 0.05$, $E_{1,j} = 0.01$, and $E_{2,j} = 0.01$, $j = 1, 2, 3$, $E(QC) = $2194.78 and ETC = $5168.42. However, when process inspection and restoration activities are carried out $E(QC)$ is reduced to $81248.7$, leading an overall cost reduction of 16%. The expected total cost is reduced to ETC = $4331.63. This is because inspection and restoration activities reduce the amount of time during which the processes remain in the out-of-control state, thus reducing quality-related costs. The increased maintenance cost is more than compensated for by the reduction in quality-related costs.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
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<td>$n$</td>
<td>3</td>
<td>$r$</td>
<td>$0.25$</td>
</tr>
<tr>
<td>$D$</td>
<td>10000</td>
<td>$\nu_j$</td>
<td>$0.05 A_j$</td>
</tr>
<tr>
<td>$P_1$</td>
<td>50000</td>
<td>$\alpha_j$</td>
<td>$0.05$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>40000</td>
<td>$\theta_j$</td>
<td>$0.05$</td>
</tr>
<tr>
<td>$P_3$</td>
<td>100000</td>
<td>$s_j$</td>
<td>$C_j$</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$100$</td>
<td>$\pi_{1,j}$</td>
<td>$0.5C_j$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$35$</td>
<td>$\pi_{2,1}$</td>
<td>$1.5C_1$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$20$</td>
<td>$\pi_{2,2}$</td>
<td>$C_1$</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$15$</td>
<td>$\pi_{2,3}$</td>
<td>$C_2$</td>
</tr>
<tr>
<td>$C_1$</td>
<td>$10$</td>
<td>$E_{1,j}$</td>
<td>$0.01$</td>
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<tr>
<td>$C_2$</td>
<td>$5$</td>
<td>$E_{2,j}$</td>
<td>$0.01$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$2$</td>
<td>$r_{0,j}$</td>
<td>$0.15A_j$</td>
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<td>$C_4$</td>
<td>$1$</td>
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</table>

Table 1. Parameter values.

<table>
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<tr>
<th>Variables</th>
<th>$Q$</th>
<th>$Q_1$</th>
<th>$Q_2$</th>
<th>$Q_3$</th>
<th>$\eta_1$</th>
<th>$\eta_2$</th>
<th>$\eta_3$</th>
<th>$E(QC)$</th>
<th>ETC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model 1</td>
<td>812</td>
<td>827</td>
<td>843</td>
<td>855</td>
<td>2194.8</td>
<td>5168.4</td>
<td></td>
<td></td>
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<tr>
<td>Model 2</td>
<td>1208</td>
<td>1223</td>
<td>1238</td>
<td>1253</td>
<td>1248.7</td>
<td>4331.6</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Optimal solutions.
In this paper, we made the assumption that non-conforming items are not produced during the in-control state of the production processes. This amounts to neglecting the falsely accepted items coming from previous stages. This assumption can be justified in practice as discussed above and was introduced in the model for mathematical simplicity. Tables 3 and 4 show that the number of falsely accepted items are indeed very small compared with the lot sizes produced. Note that these numbers can only be significant when the level of inspection errors is high, which should not be the case in practical applications.

6. Conclusions

We developed multistage lot-sizing models for imperfect production processes. The effect of inspection errors that may be committed while screening defective items that may be produced at various stages has been incorporated. In addition, inspection and restoration, if necessary, of the processes at all stages as a mean of improving quality have been considered. Numerical examples have been presented to illustrate various aspects of the model developed here. In particular, it is shown

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$Q$</th>
<th>$FA_1$</th>
<th>$Q_1$</th>
<th>$FA_2$</th>
<th>$Q_2$</th>
<th>$FA_3$</th>
<th>$Q_3$</th>
<th>$E(QC)$</th>
<th>ETC</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>812</td>
<td>0.061</td>
<td>827</td>
<td>0.077</td>
<td>843</td>
<td>0.034</td>
<td>855</td>
<td>2194.78</td>
<td>5168.42</td>
</tr>
<tr>
<td>0.10</td>
<td>628</td>
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<td>642</td>
<td>0.097</td>
<td>658</td>
<td>0.043</td>
<td>669</td>
<td>3002.51</td>
<td>6391.39</td>
</tr>
<tr>
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<td>0.084</td>
<td>422</td>
<td>0.111</td>
<td>438</td>
<td>0.048</td>
<td>447</td>
<td>4555.95</td>
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<tr>
<td>0.40</td>
<td>321</td>
<td>0.084</td>
<td>333</td>
<td>0.114</td>
<td>348</td>
<td>0.049</td>
<td>356</td>
<td>5636.89</td>
<td>11278.84</td>
</tr>
</tbody>
</table>

Effect of type II inspection error

<table>
<thead>
<tr>
<th>$E_{2,j}$</th>
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<th>$FA_1$</th>
<th>$Q_1$</th>
<th>$FA_2$</th>
<th>$Q_2$</th>
<th>$FA_3$</th>
<th>$Q_3$</th>
<th>$E(QC)$</th>
<th>ETC</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>812</td>
<td>0.031</td>
<td>827</td>
<td>0.039</td>
<td>843</td>
<td>0.017</td>
<td>855</td>
<td>2189.94</td>
<td>5163.72</td>
</tr>
<tr>
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<td>812</td>
<td>0.061</td>
<td>827</td>
<td>0.077</td>
<td>843</td>
<td>0.034</td>
<td>855</td>
<td>2194.78</td>
<td>5168.42</td>
</tr>
<tr>
<td>0.050</td>
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<td>0.302</td>
<td>820</td>
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<td>835</td>
<td>0.169</td>
<td>847</td>
<td>2222.03</td>
<td>5205.85</td>
</tr>
</tbody>
</table>

Effect of type I inspection error

<table>
<thead>
<tr>
<th>$E_{1,j}$</th>
<th>$Q$</th>
<th>$FA_1$</th>
<th>$Q_1$</th>
<th>$FA_2$</th>
<th>$Q_2$</th>
<th>$FA_3$</th>
<th>$Q_3$</th>
<th>$E(QC)$</th>
<th>ETC</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.025</td>
<td>616</td>
<td>0.075</td>
<td>639</td>
<td>0.099</td>
<td>665</td>
<td>0.045</td>
<td>687</td>
<td>4414.04</td>
<td>7845.16</td>
</tr>
<tr>
<td>0.050</td>
<td>595</td>
<td>0.074</td>
<td>634</td>
<td>0.103</td>
<td>677</td>
<td>0.049</td>
<td>718</td>
<td>6920.25</td>
<td>10428.90</td>
</tr>
<tr>
<td>0.100</td>
<td>553</td>
<td>0.071</td>
<td>622</td>
<td>0.110</td>
<td>702</td>
<td>0.059</td>
<td>786</td>
<td>12599.84</td>
<td>16288.09</td>
</tr>
</tbody>
</table>

FA$_1$, number of falsely accepted items at the end of level 1; FA$_2$, number of falsely accepted items at the end of level 2; FA$_3$, number of falsely accepted items at the end of level 3.

Table 3. Results for the model without process inspection and restoration.

In this paper, we made the assumption that non-conforming items are not produced during the in-control state of the production processes. This amounts to neglecting the falsely accepted items coming from previous stages. This assumption can be justified in practice as discussed above and was introduced in the model for mathematical simplicity. Tables 3 and 4 show that the number of falsely accepted items are indeed very small compared with the lot sizes produced. Note that these numbers can only be significant when the level of inspection errors is high, which should not be the case in practical applications.

6. Conclusions

We developed multistage lot-sizing models for imperfect production processes. The effect of inspection errors that may be committed while screening defective items that may be produced at various stages has been incorporated. In addition, inspection and restoration, if necessary, of the processes at all stages as a mean of improving quality have been considered. Numerical examples have been presented to illustrate various aspects of the model developed here. In particular, it is shown...
that lower costs are incurred when inspection and restorations are carried out because of reductions in quality-related costs.

It is believed that these models provide a step forward in the more realistic modelling and the understanding of multistage production systems. Other extensions may include investigating the effect of preventive maintenance policies on these systems. In this case, problems can be avoided before they occur, which may lead to additional gains. The effects of maintenance policies on EPQ and economic design of an x-control chart have been studied in Ben-Daya (1999) and Ben-Daya and Makhdoum (1998) in the context of single-stage production systems.

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References


